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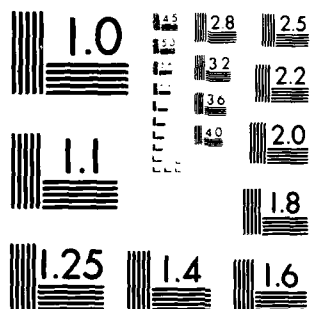UNCLASSIFIED   DAAG29-80-C-0041   F/G 12/1   NL

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THE RATE OF CONVERGENCE OF A CLASS
OF BLOCK JACOBI SCHEMES

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THE RATE OF CONVERGENCE OF A CLASS OF BLOCK JACOBI SCHEMES

Warren E. Ferguson. Jr.*

Abstract

This paper determines the rate of convergence of a class of block Jacobi iterative schemes when the schemes are applied to a general class of problems. Among these iterative schemes are the q-line and q-plane block Jacobi schemes. while the general class of problems include discretizations of elliptic and parabolic equations whose coefficients do not depend on one of the spatial variables.

AMS (MOS) Subject Classification: 65F10

Key Words: Iterative Methods, Block Jacobi Schemes, Block Gauss-Seidel Schemes, Block Successive Over-Relaxation Schemes

Work Unit Number 3 — Numerical Analysis and Scientific Computing

SIGNIFICANCE AND EXPLANATION

The solution of many elliptic and parabolic partial differential equations lead to the need to solve large linear systems. With the usual serial computer architecture point iterative schemes frequently led to the efficient solution of many of these systems. In a point iterative scheme the current estimate of the solution is improved in a repetitive fashion by modifying only one component of the solution. The advent of vector and parallel computer architectures now allow the efficient solution of these systems by using block iterative schemes. In a block iterative scheme the current estimate of the solution is improved in a repetitive fashion by modifying several components of the solution. Since point iterative schemes can be viewed as particularly simple block iterative schemes one would expect to find that block iterative schemes can potentially converge faster than point iterative schemes. In this paper the rate of convergence of one class of block iterative schemes is precisely determined.

THE RATE OF CONVERGENCE OF A CLASS OF BLOCK JACOBI SCHEMES

Warren E. Ferguson, Jr.*

1. Introduction

This paper determines the spectrum of a class of block Jacobi iterative schemes when these schemes are used to solve linear systems of the form

$$Ax = b \qquad \text{with} \qquad A = \text{trid}(-I, T, -I)$$

a square matrix of block order n. In this analysis it will be assumed that I denotes the identity matrix of order m and that T is a symmetric positive definite matrix of order m whose eigenvalues are all greater than two. Such problems arise quite naturally when elliptic and parabolic parital differential equations, whose coefficients do not depend on one of the spatial variables, are discretized. Of course, knowing the spectrum of an iterative scheme allows one to precisely describe its rate of convergence [5].

Among the block Jacobi iterative schemes considered are the q-line block Jacobi schemes and their higher dimensional variants. These schemes have been described and analyzed extensively by others [1,2,3,4]. As an application of this analysis the spectral radius of the q-line block Jacobi scheme is determined when it is applied to the two-dimensional Dirichlet version of Laplace's equation in the unit square.

To every block Jacobi scheme there is a naturally related block SOR scheme [5]. Since the block Jacobi matrices considered here are consistently ordered 2-cyclic matrices then the results of Varga can be used to determine the optimal relaxation parameter for the block SOR scheme.

In this paper block matrices will be used extensively. These block matrices all have the property that the blocks on the diagonal are themselves square matrices. Furthermore the notation diag() and trid() will be used to denote diagonal and tridiagonal matrices respectively.

2. The Block Jacobi Scheme

The block Jacobi scheme to be considered is but one member of a general class of iterative schemes which are frequently used to solve the linear system

$$Ax = b \qquad (2.1a)$$

This general class of iterative schemes has the form

$$Bx^{(\nu)} = Cx^{(\nu-1)} - b \qquad \text{for} \qquad \nu = 1, 2, 3, \dots \qquad (2.1b)$$

where $x^{(0)}$ is an initial approximation to x while B and C describe a splitting of A[5]; i.e. B is nonsingular and

$$A = B - C. \tag{2.1c}$$

It is well-known that the rate of convergence of the iterative scheme (2.1b) is described by the root $\lambda = \rho$ of

$$|\lambda B - C| = 0 \tag{2.1d}$$

having largest magnitude[5]. In equation (2.1d) the function $|\cdot|$ represents the determinant function. Clearly (2.1d) is a generalized eigenvalue problem.

In this analysis it will be assumed that

$$A = \operatorname{trid}(-I, T, -I) \tag{2.2a}$$

is a square matrix of block order n where the identity matrix I and T are square matrices of order m. The matrix T is also assumed to be a symmetric positive definite matrix whose eigenvalues are all greater than two. Since T is a symmetric matrix then there is an orthogonal matrix P which diagonalizes T; i.e.

$$P'TP = \operatorname{diag}(t_1, t_2, \ldots, t_m) \tag{2.2b}$$

where P' denotes the transpose of P. Without loss of generality it will be supposed that the eigenvalues $t_1, t_2, \ldots, t_m$ of T are ordered so that

$$2 < t_1 \leq t_2 \leq \ldots \leq t_m. \tag{2.2c}$$

Consider the case $n = pq$ where $p \geq 2$ and $q \geq 1$ are integers. The block Jacobi scheme to be considered is of the form (2.1b) with

$$B = \operatorname{trid}(0, M, 0) \quad \text{and} \quad C = \operatorname{trid}(N, 0, N'). \tag{2.3a}$$

both square matrices of block order p where

$$M = \operatorname{trid}(-I, T, -I) \quad \text{and} \quad N = \begin{bmatrix} 0 & \ldots & 0 & I \\ 0 & \ldots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{bmatrix} \tag{2.3b}$$

are themselves square matrices of block order q. Since B is a symmetric positive definite matrix it follows that the roots of (2.1d) are real. It will be assumed that $q \geq 2$ since the roots of (2.1d) when $q = 1$ have already been determined by Arms, Gates and Zondek[1].

3. Simplification of the Generalized Eigenvalue Problem

One of the basic tools used to determine the roots of (2.1d) is the following explicit representation

$$M_{i,j}^{-1} = \begin{cases} S_q^{-1}(T) S_{i-1}(T) S_{q-j}(T), & \text{when } i \leq j; \\ S_q^{-1}(T) S_{j-1}(T) S_{q-i}(T), & \text{when } i \geq j \end{cases} \tag{3.1a}$$

of $M^{-1} = (M_{i,j}^{-1})$ due to Bank[6]. Here $S_l(x)$ denotes the lth degree Chebyshev polynomial of the first kind scaled for the interval $[0,2]$. Since the eigenvalues of T are greater than two then

$$S_l(x) = \frac{\sinh\left((l+1)\phi\right)}{\sinh(\phi)} \qquad \text{with} \qquad \cosh(\phi) = \frac{x}{2} \tag{3.1b}$$

is an appropriate representation of these polynomials. Based on this representation of M^{-1} it follows that (2.1d) can be reduced to the equation

$$|\lambda I - D| = 0 \tag{3.1c}$$

where

$$
\begin{aligned}
D &\equiv B^{-1}C = \text{Trid}(M^{-1}N, 0, M^{-1}N'). \\
M^{-1}N &= [0, \ldots, 0, M_{-,1}^{-1}], \text{ and} \\
M^{-1}N' &= [M_{-,q}^{-1}, 0, \ldots, 0].
\end{aligned}
\tag{3.1d}
$$

Here $M_{-,1}^{-1}$ and $M_{-,q}^{-1}$ denote respectively the first and last block columns of M^{-1}.

The further simplification of equation (2.4c) is best explained by considering the special case $p = 3$. In this case $\lambda I - D$ has the following nonzero structure

$$
\begin{array}{ccccccccc}
(1) & (2) & (3) & & (3) & (2) & (3) & & (3) & (2) & (1)
\end{array}
$$

$$
\left(
\begin{array}{ccc:ccc:ccc}
\lambda I & & & & -F & & & & \\
& \lambda I & & & \vdots & & & & \\
& & \lambda I & & -G & & & & \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
& & -G & & \lambda I & & & & -F \\
& & \vdots & & & \lambda I & & & \vdots \\
& & -F & & & & \lambda I & & -G \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
& & \vdots & & & & -G & & \lambda I \\
& & \vdots & & & & \vdots & & & \lambda I \\
& & \vdots & & & & -F & & & & \lambda I
\end{array}
\right)
\tag{3.2a}
$$

where

$$F = S_q^{-1}(T) \qquad \text{and} \qquad G = S_q^{-1}(T)S_{q-1}(T) \tag{3.2b}$$

By expanding the determinant of the matrix described by (3.2a) columnwise along the block columns labeled (1) and (2), then interchanging the adjacent block columns labeled by (3), it follows in the general case that (3.1c) simplifies to

$$\lambda^{[n-2(p-1)]m} |E| = 0 \tag{3.2c}$$

where

$$E = \text{trid}\left(\begin{bmatrix} 0 & -F \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -G & \lambda I \\ \lambda I & -G \end{bmatrix}, \begin{bmatrix} 0 & -F \\ 0 & 0 \end{bmatrix}' \right) \tag{3.2d}$$

is a square matrix of block order $p - 1$. The analysis now continues by determining the roots of

$$|E| = 0. \tag{3.2e}$$

Since P diagonalizes T it also diagonalizes both F and G. Indeed

$$P'FP = \text{diag}(f_1, f_2, \ldots, f_m), \quad \text{and} \quad P'GP = \text{diag}(g_1, g_2, \ldots, g_m) \tag{3.3a}$$

where

$$f_l = S_q^{-1}(t_l) \quad \text{and} \quad g_l = S_q^{-1}(t_l)S_{q-1}(t_l) \quad \text{for} \quad l = 1, 2, \ldots, m. \tag{3.3b}$$

Next define

$$Q \equiv \text{diag}(P, P, \ldots, P) \tag{3.3c}$$

as a square matrix of block order $2(p - 1)$ and note that the graph of $Q'EQ$ consists of exactly m connected components, each component connecting columns of $Q'EQ$ whose indices modulo m are the same. From this fact it follows that

$$|E| = |E_1| |E_2| \ldots |E_m| \tag{3.3d}$$

where

$$E_l = \text{trid}\left(\begin{bmatrix} 0 & -f_l \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -g_l & \lambda \\ \lambda & -g_l \end{bmatrix}, \begin{bmatrix} 0 & -f_l \\ 0 & 0 \end{bmatrix}' \right) \quad \text{for} \quad l = 1, 2, \ldots, m. \tag{3.3e}$$

is a square matrix of block order $p - 1$ for $l = 1, 2, \ldots, m$. Therefore the roots of (3.2e) can be described as the union of the roots of

$$|E_l| = 0 \quad \text{for} \quad l = 1, 2, \ldots, m \tag{3.3f}$$

Since each of the equations in (3.3f) has the same general form it would seem wise to characterize the roots of the *generic equation*

$$|H_k| = 0 \tag{3.3g}$$

where

$$H_k = \text{trid}\left(\begin{bmatrix} 0 & -f \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -g & \lambda \\ \lambda & -g \end{bmatrix}, \begin{bmatrix} 0 & -f \\ 0 & 0 \end{bmatrix}' \right) \tag{3.3h}$$

is a square matrix of block order k with

$$0 < f < g. \tag{3.3i}$$

The assumption (3.3i) is justified in view of equations (2.2c), (3.1b) and (3.3b).

4. Solution of the Generic Equation

The first step in the determination of the roots of the generic equation (3.3g) is to realize that the matrix described by (3.3h) is in actuality a tridiagonal matrix. If $\tilde{H}_k$ denotes the matrix obtained by deleting the last row and column from H_k it follows that the determinants of H_k and $\tilde{H}_k$ satisfy the recurrence relations

$$|H_k| = -\lambda^2 |H_{k-1}| - g |\tilde{H}_k| , \quad \text{and}$$

$$|\tilde{H}_k| = -f^2 |\tilde{H}_{k-1}| - g |H_{k-1}| . \tag{4.1a}$$

Consequently it is easily shown that

$$\begin{bmatrix} |H_k| \\ |\tilde{H}_k| \end{bmatrix} = \begin{bmatrix} g^2 - \lambda^2 & gf^2 \\ -g & -f^2 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{4.1b}$$

Since $|H_k|$ depends on λ^2 it follows that the roots of (3.3g) occur, as expected, in positive/negative pairs.

The next step in the determination of the roots of the generic equation (3.3g) consists of the localization of its roots. Recall that one implication of Gerschgorin's theorem is that irreducibly diagonally dominant matrices are never singular[5]. By applying this result to the matrix H_k in (3.3h) and its column permuted form

$$\text{trid}\left(\begin{bmatrix} -f & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & -g \\ -g & \lambda \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -f \end{bmatrix} \right) \tag{4.2a}$$

it follows that the roots of (3.3g) lie either in the open subinterval $(g - f, g + f)$ of the positive real line or its negative image $(-g - f, -g + f)$.

Assume that λ lies in the open subinterval $(g - f, g + f)$ of the positive real line. For these values of λ the two eigenvalues μ_1 and μ_2 of the matrix

$$\begin{bmatrix} g^2 - \lambda^2 & gf^2 \\ -g & -f^2 \end{bmatrix} \tag{4.3a}$$

are distinct and admit the representation

$$\mu_1 = \lambda f e^{i\theta} \quad \text{and} \quad \mu_2 = \lambda f e^{-i\theta} \tag{4.3b}$$

where θ is that angle in the open inverval $(0, \pi)$ defined by the equation

$$\cos(\theta) = \frac{g^2 - \lambda^2 - f^2}{2\lambda f} . \tag{4.3c}$$

Note that θ increases from 0 to π as λ increases from $g - f$ to $g + f$. The matrix

$$\begin{bmatrix} \mu_1 + f^2 & \mu_2 + f^2 \\ -g & -g \end{bmatrix} \tag{4.3d}$$

diagonalizes the matrix in (4 3a) and so a short calculation shows that

$$
\begin{bmatrix} |H_k| \\ |\hat{H}_k| \end{bmatrix} = \frac{1}{\mu_1 - \mu_2} \begin{bmatrix} (\mu_1 + f^2)\mu_1^k - (\mu_2 + f^2)\mu_2^k \\ -g(\mu_1^k - \mu_2^k) \end{bmatrix}
\tag{4.3e}
$$

From the representation

$$
S_l(2\cos(\theta)) = \frac{\sin(l-1\,\theta)}{\sin(\theta)} \qquad \text{for} \qquad 0 < \theta < \pi
\tag{4.3f}
$$

of the Chebyshev polynomials of the first kind (4.3e) can also be written as

$$
\begin{bmatrix} |H_k| \\ |\hat{H}_k| \end{bmatrix} = (\lambda f)^{k-1} \begin{bmatrix} (\lambda f)S_k\left(\dfrac{g^2 - \lambda^2 - f^2}{\lambda f}\right) - f^2 S_{k-1}\left(\dfrac{g^2 - \lambda^2 - f^2}{\lambda f}\right) \\ -g S_{k-1}\left(\dfrac{g^2 - \lambda^2 - f^2}{\lambda f}\right) \end{bmatrix}.
\tag{4.3g}
$$

It follows that (4.3g) holds for all λ since both sides are polynomials in λ which agree in value for all λ in the nonempty open interval $(g - f, g - f)$.

It is now a simple matter to describe the roots of the generic equation (3.3g). The roots of (3.3g) occur in positive/negative pairs with the positive roots admitting the representation

$$
\lambda = -f\cos(\theta) + \sqrt{g^2 - f^2\sin^2(\theta)}
\tag{4.4a}
$$

where θ is any one of the k distinct roots of

$$
\tan((k+1)\theta) = \frac{f\sin(\theta)}{\sqrt{g^2 - f^2\sin^2(\theta)}}
\tag{4.4b}
$$

in the open interval $(0, \pi)$. As illustrated by Figure 1 when $k = 3$ and $g = 1.1f$, there is exactly one root of (4.4b) in each of the disjoint intervals

$$
J_i = \left(\frac{i - \frac{1}{2}}{(k+1)}\pi, \frac{i}{(k+1)}\pi\right) \qquad \text{for} \quad i = 1, 2, \ldots, k.
\tag{4.4c}
$$

Note that as $k \to \infty$ the intervals $J_1, J_2, \ldots, J_k$ densely fill the interval $(0, \pi)$ with the points in J_k uniformly approaching π. Consequently the sharpest upper bound for the largest root of (3.3g) that is valid for all k is

$$
\lambda \sim g + f.
\tag{4.4d}
$$

Note that this is the same upper bound on the roots of (3.3g) described by Gerschgorin's theorem.

5 The Spectrum of the Block Jacobi Scheme

Using the results of the previous section the specturm of the block Jacobi scheme can be described as follows. As shown by (3.2c). exactly $n - 2(p-1)m$ roots of (2.1d) are zero. Of the remaining $2(p-1)m$ nonzero roots of (2.1d) half of the roots are positive and the other half are the negative images of the positive roots The $(p-1)m$ positive roots can be divided into m clusters
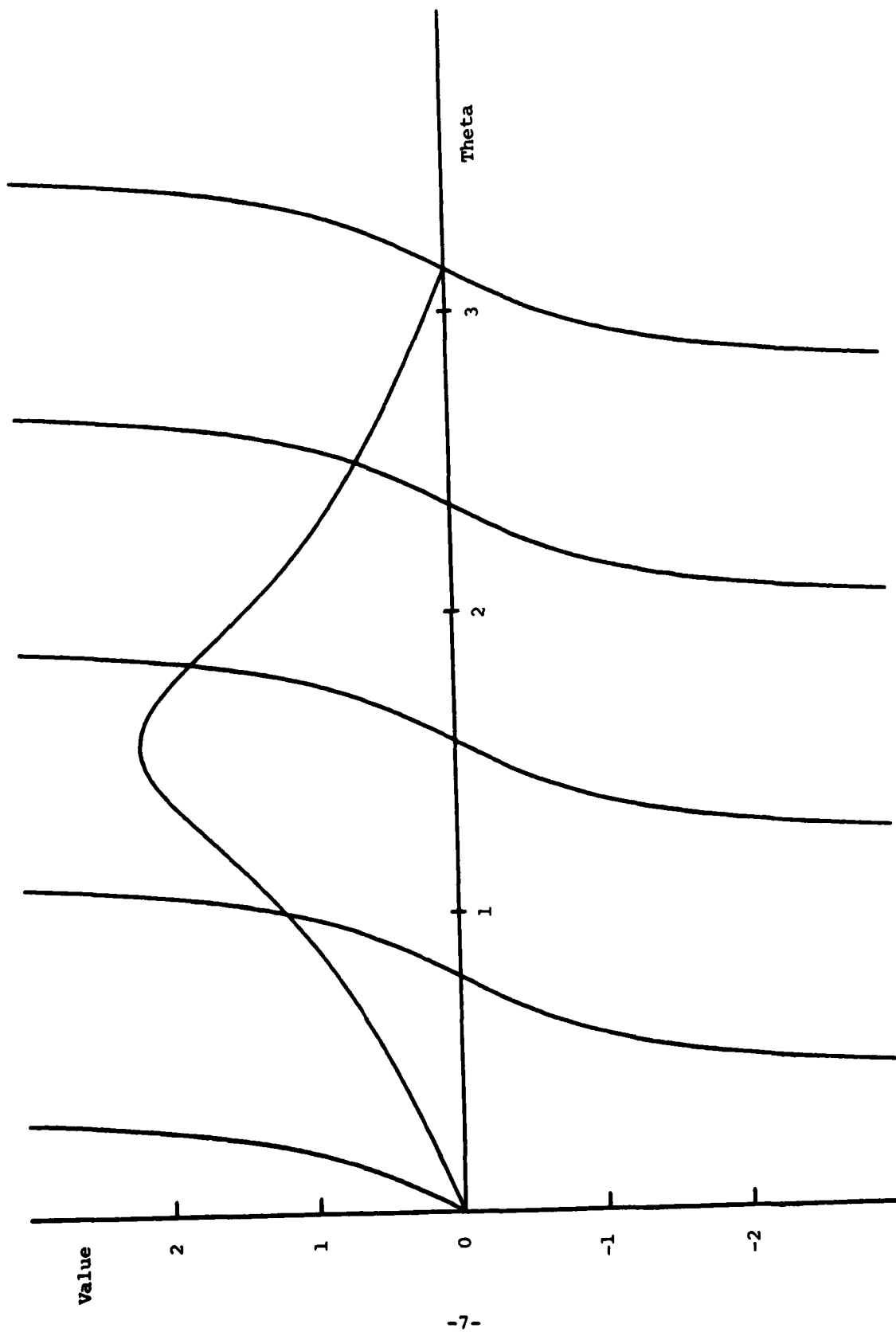
FIGURE 1. Graphical Interpretation of Equation (4.4b) when k = 3 and g = 1.1f.

each containing $p - 1$ roots. The lth such cluster consists of the roots of (2.1d) lying in the open interval $(g_l - f_l, g_l + f_l)$ and can be obtained from (4.4a) by solving (4.4b) with $k = p - 1$, $f = f_l$ and $g = g_l$ for the $p - 1$ roots θ lying in $(0, \pi)$.

The clusters described above are also the same clusters predicted and observed by Kratzer. Parter, and Steuerwalt [4]. Note that the radius f_l of the lth cluster decreases rapidly as q increases.

Since f_l and g_l are decreasing functions of l it follows that the root ρ of (2.1d) having largest magnitude belongs to the cluster of roots lying in $(g_1 - f_1, g_1 - f_1)$. It can be obtained from (4.4a) by solving (4.4b) with $k = p - 1$, $f = f_1$ and $g = g_1$ for the largest root θ lying in $(0, \pi)$. Of course, knowledge of ρ allows one to determine the optimal relaxation parameter ω for the block SOR methods naturally related to this block Jacobi scheme [5]. Since the iteration matrix for this block Jacobi scheme is a consistently ordered 2-cyclic block matrix it follows that

$$\omega = \frac{2}{1 + \sqrt{1 - \rho^2}} \tag{5.1a}$$

is the optimal relaxation parameter.

As stated in the previous section, the sharpest upper bound on ρ which is valid for all p is the bound

$$\rho \sim f_1 + g_1 . \tag{5.2a}$$

When

$$T = \text{trid}(-1, 4, -1) \tag{5, 2b}$$

is a square matrix of order m then it is well-known that

$$t_1 = 2 + 4 \sin^2 \left(\frac{1}{2} \pi h \right) \qquad \text{where} \qquad h \equiv \frac{1}{m + 1} . \tag{5.2c}$$

After a straightforward Taylor series expansion it follows that

$$\rho \sim 1 - \frac{1}{2} q (\pi h)^2 . \tag{5.2c}$$

a well-known result of Parter [2].

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